$SPAD/src/input westeralgebra.input

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Abstract

These problems come from the web page

`http://math.unm.edu/~wester/cas_review.html`

Contents

```
)set break resume
)set messages autoload off
)set streams calculate 7
)sys rm -f westeralgebra.output
)spool westeralgebra.output
)clear all
```

————-

1 Algebra

One would think that the simplification $2\ 2^n => 2^{(n+1)}$ would happen automatically or at least easily ...

—— * ——

```
--S 1 of 63
2*2**n
--R
--R
--R           n
--R   (1)  2 2
--R                                        Type: Expression(Integer)
--E 1
```

————-

And how about $4\ 2^n => 2^{(n+2)}$? [Richard Fateman]

—— * ——

```
--S 2 of 63
4*2**n
--R
--R
--R           n
--R   (2)  4 2
--R                                        Type: Expression(Integer)
--E 2
```

————-

$(-1)^{(n(n+1))} => 1$ for integer n

—— * ——

```
--S 3 of 63
(-1)**(n*(n + 1))
--R
```

```
--R
--R                  2
--R               n  + n
--R    (3)  (- 1)
--R                                                    Type: Expression(Integer)
--E 3
```

Also easy $=> 2(3x - 5)$
— * —

```
--S 4 of 63
factor(6*x - 10)
--R
--R
--R    (4)  2(3x - 5)
--R                                          Type: Factored(Polynomial(Integer))
--E 4
```

Univariate gcd: $gcd(p1, p2) => 1$, $gcd(p1q, p2q) => q$ [Richard Liska]
— * —

```
--S 5 of 63
p1:= 64*x**34 - 21*x**47 - 126*x**8 - 46*x**5 - 16*x**60 - 81
--R
--R
--R              60      47       34       8        5
--R    (5)  - 16x    - 21x    + 64x    - 126x   - 46x   - 81
--R                                              Type: Polynomial(Integer)
--E 5
```

```
--S 6 of 63
p2:= 72*x**60 - 25*x**25 - 19*x**23 - 22*x**39 - 83*x**52 + 54*x**10 + 81
--R
--R
--R             60       52       39       25       23        10
--R    (6)  72x    - 83x    - 22x    - 25x    - 19x    + 54x    + 81
--R                                              Type: Polynomial(Integer)
--E 6
```

```
--S 7 of 63
q:= 34*x**19 - 25*x**16 + 70*x**7 + 20*x**3 - 91*x - 86
--R
--R
--R             19       16      7       3
--R    (7)  34x    - 25x    + 70x  + 20x  - 91x - 86
--R                                              Type: Polynomial(Integer)
--E 7
```

4

```
--S 8 of 63
gcd(p1, p2)
--R
--R
--R   (8)  1
--R                                              Type: Polynomial(Integer)
--E 8

--S 9 of 63
gcd(expand(p1*q), expand(p2*q)) - q
--R
--R
--R   (9)  0
--R                                              Type: Polynomial(Integer)
--E 9
```

─────-
$resultant(p1q, p2q) => 0$
── * ──

```
--S 10 of 63
resultant(expand(p1*q), expand(p2*q), x)
--R
--R
--R   (10)  0
--R                                              Type: Polynomial(Integer)
--E 10
```

─────-
How about factorization? $=> p1 * p2$
── * ──

```
--S 11 of 63
factor(expand(p1 * p2))
--R
--R
--R   (11)
--R     -
--R          60      47      34       8      5
--R      (16x   + 21x  - 64x   + 126x + 46x  + 81)
--R   *
--R          60      52      39      25      23      10
--R      (72x   - 83x  - 22x   - 25x   - 19x   + 54x   + 81)
--R                                Type: Factored(Polynomial(Integer))
--E 11

)clear properties p1 p2 q
```

———————-

Multivariate gcd: $gcd(p1, p2) => 1, gcd(p1q, p2q) => q$

—— * ——

```
--S 12 of 63
p1:= 24*x*y**19*z**8 - 47*x**17*y**5*z**8 + 6*x**15*y**9*z**2 - 3*x**22 + 5
--R
--R
--R             19       17 5  8     15 9 2     22
--R   (12)  (24x y   - 47x  y )z  + 6x  y z  - 3x   + 5
--R                                      Type: Polynomial(Integer)
--E 12


--S 13 of 63
p2:= 34*x**5*y**8*z**13 + 20*x**7*y**7*z**7 + 12*x**9*y**16*z**4 + 80*y**14*z
--R
--R
--R            5 8 13      7 7 7      9 16 4      14
--R   (13)  34x y z    + 20x y z  + 12x y  z  + 80y  z
--R                                      Type: Polynomial(Integer)
--E 13


--S 14 of 63
q:= 11*x**12*y**7*z**13 - 23*x**2*y**8*z**10 + 47*x**17*y**5*z**8
--R
--R
--R            12 7 13      2 8 10      17 5 8
--R   (14)  11x  y z    - 23x y z   + 47x  y z
--R                                      Type: Polynomial(Integer)
--E 14


--S 15 of 63
gcd(p1, p2)
--R
--R
--R   (15)  1
--R                                      Type: Polynomial(Integer)
--E 15


--S 16 of 63
gcd(expand(p1*q), expand(p2*q)) - q
--R
--R
--R   (16)  0
--R                                      Type: Polynomial(Integer)
--E 16
```

———————-

How about factorization? $=> p1 * p2$
          — * —

```
--S 17 of 63
factor(expand(p1 * p2))
--R
--R
--R   (17)
--R        7          19        17 5  8      15 9 2       22
--R     2y z((24x  y    - 47x   y )z  + 6x   y z  - 3x    + 5)
--R   *
--R          5    12       7 6      9 9 3      7
--R     (17x  y  z    + 10x z  + 6x  y z  + 40y )
--R                                   Type: Factored(Polynomial(Integer))
--E 17

)clear properties p1 p2 q
```

————-

$=> x^n \ for \ n > 0$ [Chris Hurlburt]
          — * —

```
--S 18 of 63
gcd(2*x**(n + 4) - x**(n + 2), 4*x**(n + 1) + 3*x**n)
--R
--R
--R   (18)  1
--R                                   Type: Expression(Integer)
--E 18
```

————-

Resultants. If the resultant of two polynomials is zero, this implies they have a common factor. See Keith O. Geddes, Stephen R. Czapor and George Labahn, "Algorithms for Computer Algebra", Kluwer Academic Publishers, 1992, p. 286 $=> 0$
          — * —

```
--S 19 of 63
resultant(3*x**4 + 3*x**3 + x**2 - x - 2, x**3 - 3*x**2 + x + 5, x)
--R
--R
--R   (19)  0
--R                                   Type: Polynomial(Integer)
--E 19
```

————-

Numbers are nice, but symbols allow for variability—try some high school algebra: rational simplification $=> (x-2)/(x+2)$
          — * —

```
--S 20 of 63
(x**2 - 4)/(x**2 + 4*x + 4)
--R
--R
--R        x - 2
--R   (20)  -----
--R        x + 2
--R                                        Type: Fraction(Polynomial(Integer))
--E 20
```

———-

This example requires more sophistication $=> e^{(x/2)} - 1$

— * —

```
--S 21 of 63
[(%e**x - 1)/(%e**(x/2) + 1), (exp(x) - 1)/(exp(x/2) + 1)]
--R
--R
--R            x       x
--R         %e  - 1 %e  - 1
--R   (21)  [-------,-------]
--R            x       x
--R            -       -
--R            2       2
--R         %e  + 1 %e  + 1
--R                                        Type: List(Expression(Integer))
--E 21
```

```
--S 22 of 63
map(normalize, %)
--R
--R
--R            x       x
--R            -       -
--R            2       2
--R   (22)  [%e  - 1,%e  - 1]
--R                                        Type: List(Expression(Integer))
--E 22
```

———-

Expand and factor polynomials

— * —

```
--S 23 of 63
(x + 1)**20
--R
--R
--R   (23)
--R        20        19        18        17        16        15        14        13
```

```
--R      x     + 20x     + 190x     + 1140x    + 4845x    + 15504x    + 38760x    + 77520x
--R    +
--R             12           11            10          9           8          7           6
--R      125970x    + 167960x    + 184756x    + 167960x    + 125970x   + 77520x   + 38760x
--R    +
--R             5        4         3        2
--R      15504x   + 4845x   + 1140x   + 190x   + 20x + 1
--R                                                      Type: Polynomial(Integer)
--E 23


--S 24 of 63
D(%, x)
--R
--R
--R    (24)
--R          19         18          17           16          15           14           13
--R      20x     + 380x     + 3420x    + 19380x    + 77520x    + 232560x    + 542640x
--R    +
--R              12            11            10          9            8             7
--R      1007760x    + 1511640x    + 1847560x    + 1847560x   + 1511640x    + 1007760x
--R    +
--R              6          5          4          3         2
--R      542640x   + 232560x   + 77520x   + 19380x   + 3420x   + 380x + 20
--R                                                      Type: Polynomial(Integer)
--E 24


--S 25 of 63
factor(%)
--R
--R
--R                     19
--R    (25)  20(x + 1)
--R                                              Type: Factored(Polynomial(Integer))
--E 25


                ————-
```

Completely factor this polynomial, then try to multiply it back together!
                —— * ——

```
--S 26 of 63
radicalSolve(x**3 + x**2 - 7 = 0, x)
--R
--R
--R    (26)
--R    [
--R      x =
--R                                  +------------------+2
--R                                  |   +----+        +-+
--R                  +---+           |9\|1295  + 187\|3
```

```
--R                (- 9\|- 3  + 9)  |------------------
--R                                3|          +-+
--R                                \|      54\|3
--R           +
--R                                   +-----------------+
--R                                   | +----+      +-+
--R               +---+           |9\|1295  + 187\|3
--R            (- 3\|- 3  - 3)  |----------------- - 2
--R                                3|          +-+
--R                                \|      54\|3
--R       /
--R                                   +-----------------+
--R                                   | +----+      +-+
--R               +---+           |9\|1295  + 187\|3
--R            (9\|- 3  + 9)  |-----------------
--R                                3|          +-+
--R                                \|      54\|3
--R        ,
--R
--R      x =
--R                                   +----------------+2
--R                                   | +----+      +-+
--R               +---+           |9\|1295  + 187\|3
--R            (- 9\|- 3  - 9)  |------------------
--R                                3|            +-+
--R                                \|      54\|3
--R           +
--R                                   +-----------------+
--R                                   | +----+      +-+
--R               +---+           |9\|1295  + 187\|3
--R            (- 3\|- 3  + 3)  |----------------- + 2
--R                                3|          +-+
--R                                \|      54\|3
--R       /
--R                                   +-----------------+
--R                                   | +----+      +-+
--R               +---+           |9\|1295  + 187\|3
--R            (9\|- 3  - 9)  |-----------------
--R                                3|          +-+
--R                                \|      54\|3
--R        ,
--R           +----------------+2    +-----------------+
--R           | +----+      +-+      | +----+      +-+
--R           |9\|1295  + 187\|3     |9\|1295  + 187\|3
--R         9 |-----------------  - 3 |----------------- + 1
--R           3|          +-+        3|          +-+
--R           \|      54\|3           \|      54\|3
--R     x= -------------------------------------------------]
--R                                   +----------------+
--R                                   | +----+        +-+
```

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```
--R                       |9\|1295  + 187\|3
--R                 9  |------------------
--R                 3|          +-+
--R                 \|      54\|3
--R                              Type: List(Equation(Expression(Integer)))
--E 26


--S 27 of 63
reduce(*, map(e +-> lhs(e) - rhs(e), %))
--R
--R
--R           3    2       +-+ +----+       3     2
--R        (9x  + 9x  - 63)\|3 \|1295  + 561x  + 561x  - 3927
--R   (27)  ---------------------------------------------
--R                       +-+ +----+       +-+2
--R                     9\|3 \|1295  + 187\|3
--R                                           Type: Expression(Integer)
--E 27


--S 28 of 63
x**100 - 1
--R
--R
--R          100
--R   (28)  x    - 1
--R                                           Type: Polynomial(Integer)
--E 28


--S 29 of 63
factor(%)
--R
--R
--R   (29)
--R                  2      4    3    2          4    3    2
--R     (x - 1)(x + 1)(x  + 1)(x  - x  + x  - x + 1)(x  + x  + x  + x + 1)
--R   *
--R      8    6    4    2      20    15    10    5       20    15    10    5
--R     (x  - x  + x  - x  + 1)(x   - x   + x   - x  + 1)(x   + x   + x   + x  + 1)
--R   *
--R       40    30    20    10
--R     (x   - x   + x   - x   + 1)
--R                                           Type: Factored(Polynomial(Integer))
--E 29
```

———-

Factorization over the complex rationals
$$=> (2x + 3i)(2x - 3i)(x + 1 + 4i)(x + 1 - 4i)$$
— * —

```
--S 30 of 63
```

```
factor(4*x**4 + 8*x**3 + 77*x**2 + 18*x + 153, [rootOf(i**2 + 1)])
--R
--R
--R                           3i        3i
--R   (30)   4(x - 4i + 1)(x - --)(x + --)(x + 4i + 1)
--R                            2         2
--R                                      Type: Factored(Polynomial(AlgebraicNumber))
--E 30
```

———-
Algebraic extensions
___ * ___

```
--S 31 of 63
sqrt2:= rootOf(sqrt2**2 - 2)
--R
--R
--R   (31)  sqrt2
--R                                                  Type: AlgebraicNumber
--E 31
```

———-
$=> sqrt2 + 1$
___ * ___

```
--S 32 of 63
1/(sqrt2 - 1)
--R
--R
--R   (32)  sqrt2 + 1
--R                                                  Type: AlgebraicNumber
--E 32
```

———-
$=> (x^2 - 2x - 3)/(x - sqrt2) = (x + 1)(x - 3)/(x - sqrt2)$ [Richard Liska]
___ * ___

```
--S 33 of 63
(x**3 + (sqrt2 - 2)*x**2 - (2*sqrt2 + 3)*x - 3*sqrt2)/(x**2 - 2)
--R
--R
--R            2
--R          x  - 2x - 3
--R   (33)  -----------
--R           x - sqrt2
--R                                      Type: Fraction(Polynomial(AlgebraicNumber))
--E 33
```

```
--S 34 of 63
numer(%)/ratDenom(denom(%))
--R
--R
--R              2
--R          - x  + 2x + 3
--R   (34)  -------------
--R           sqrt2 - x
--R                                              Type: Expression(Integer)
--E 34

)clear properties sqrt2
```

————-

Multiple algebraic extensions
— * —

```
--S 35 of 63
sqrt3:= rootOf(sqrt3**2 - 3)
--R
--R
--R   (35)  sqrt3
--R                                                 Type: AlgebraicNumber
--E 35

--S 36 of 63
cbrt2:= rootOf(cbrt2**3 - 2)
--R
--R
--R   (36)  cbrt2
--R                                                 Type: AlgebraicNumber
--E 36
```

————-

$=> 2cbrt2 + 8sqrt3 + 18cbrt2^2 + 12cbrt2sqrt3 + 9$
— * —

```
--S 37 of 63
(cbrt2 + sqrt3)**4
--R
--R
--R                                     2
--R   (37)  (12cbrt2 + 8)sqrt3 + 18cbrt2  + 2cbrt2 + 9
--R                                                 Type: AlgebraicNumber
--E 37

)clear properties sqrt3 cbrt2
```

————-
Factor polynomials over finite fields and field extensions
— * —

```
--S 38 of 63
p:= x**4 - 3*x**2 + 1
--R
--R
--R          4     2
--R   (38)  x  - 3x  + 1
--R                                        Type: Polynomial(Integer)
--E 38

--S 39 of 63
factor(p)
--R
--R
--R          2             2
--R   (39)  (x  - x - 1)(x  + x - 1)
--R                                   Type: Factored(Polynomial(Integer))
--E 39
```

————-
$=> (x-2)^2(x+2)^2 \ mod \ 5$
— * —

```
--S 40 of 63
factor(p :: Polynomial(PrimeField(5)))
--R
--R
--R              2       2
--R   (40)  (x + 2) (x + 3)
--R                              Type: Factored(Polynomial(PrimeField(5)))
--E 40

--S 41 of 63
expand(%)
--R
--R
--R          4     2
--R   (41)  x  + 2x  + 1
--R                                     Type: Polynomial(PrimeField(5))
--E 41
```

————-
$=> (x^2+x+1)(x^9-x^8+x^6-x^5+x^3-x^2+1) \ mod \ 65537$ [Paul Zimmermann]
— * —

```
--S 42 of 63
```

```
factor(x**11 + x + 1 :: Polynomial(PrimeField(65537)))
--R
--R
--R          2          9        8   6         5   3         2
--R  (42)  (x  + x + 1)(x  + 65536x  + x  + 65536x  + x  + 65536x  + 1)
--R                              Type: Factored(Polynomial(PrimeField(65537)))
--E 42
```

$$\overline{}-$$
$$=> (x - phi)(x + phi)(x - phi + 1)(x + phi - 1)$$
$$\text{where } phi^2 - phi - 1 = 0 \text{ or } phi = (1 \pm sqrt(5))/2$$
$$-\!\!\!-\ *\ -\!\!\!-$$

```
--S 43 of 63
phi:= rootOf(phi**2 - phi - 1)
--R
--R
--R   (43)  phi
--R                                              Type: AlgebraicNumber
--E 43

--S 44 of 63
factor(p, [phi])
--R
--R
--R   (44)  (x - phi)(x - phi + 1)(x + phi - 1)(x + phi)
--R                              Type: Factored(Polynomial(AlgebraicNumber))
--E 44

)clear properties phi p

--S 45 of 63
expand((x - 2*y**2 + 3*z**3)**20)
--R
--R
--R   (45)
--R               60                 2                   57
--R    3486784401z   + (- 46490458680y  + 23245229340x)z
--R   +
--R                4                 2            2 54
--R    (294439571640y  - 294439571640x y  + 73609892910x )z
--R   +
--R                          6                 4                2 2
--R           - 1177758286560y  + 1766637429840x y  - 883318714920x y
--R         +
--R                          3
--R           147219785820x
--R   *
--R        51
```

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```
--R        z
--R   +
--R                       8                  6                    2 4
--R           3336981811920y  - 6673963623840x y  + 5005472717880x y
--R       +
--R                      3 2               4
--R           - 1668490905960x y  + 208561363245x
--R     *
--R          48
--R        z
--R   +
--R                        10                 8                    2 6
--R           - 7118894532096y    + 17797236330240x y  - 17797236330240x y
--R       +
--R                      3 4                 4 2                 5
--R           8898618165120x y  - 2224654541280x y  + 222465454128x
--R     *
--R          45
--R        z
--R   +
--R                        12                  10                   2 8
--R           11864824220160y  - 35594472660480x y    + 44493090825600x y
--R       +
--R                      3 6                 4 4                 5 2
--R           - 29662060550400x y  + 11123272706400x y  - 2224654541280x y
--R       +
--R                      6
--R           185387878440x
--R     *
--R          42
--R        z
--R   +
--R                        14                  12                   2 10
--R           - 15819765626880y    + 55369179694080x y    - 83053769541120x y
--R       +
--R                      3 8                 4 6                 5 4
--R           69211474617600x y  - 34605737308800x y  + 10381721192640x y
--R       +
--R                      6 2               7
--R           - 1730286865440x y  + 123591918960x
--R     *
--R          39
--R        z
--R   +
--R                        16                  14                  2 12
--R           17138079429120y  - 68552317716480x y    + 119966556003840x y
--R       +
--R                      3 10                4 8                  5 6
--R           - 119966556003840x y   + 74979097502400x y  - 29991639000960x y
--R       +
```

```
--R                          6 4                       7 2                   8
--R          7497909750240x y  - 1071129964320x y  + 66945622770x
--R    *
--R          36
--R         z
--R    +
--R                             18                      16                      2 14
--R           - 15233848381440y    + 68552317716480x y    - 137104635432960x y
--R        +
--R                           3 12                   4 10                    5 8
--R           159955408005120x y    - 119966556003840x y    + 59983278001920x y
--R        +
--R                           6 6                     7 4                   8 2
--R           - 19994426000640x y   + 4284519857280x y   - 535564982160x y
--R        +
--R                       9
--R           29753610120x
--R    *
--R          33
--R         z
--R    +
--R                              20                      18                    2 16
--R           11171488813056y   - 55857444065280x y    + 125679249146880x y
--R        +
--R                            3 14                     4 12                      5 10
--R           - 167572332195840x y    + 146625790671360x y    - 87975474402816x y
--R        +
--R                          6 8                    7 6                   8 4
--R           36656447667840x y  - 10473270762240x y  + 1963738267920x y
--R        +
--R                         9 2                  10
--R           - 218193140880x y   + 10909657044x
--R    *
--R          30
--R         z
--R    +
--R                             22                      20                    2 18
--R           - 6770599280640y    + 37238296043520x y    - 93095740108800x y
--R        +
--R                           3 16                      4 14                     5 12
--R           139643610163200x y    - 139643610163200x y    + 97750527114240x y
--R        +
--R                           6 10                    7 8                    8 6
--R           - 48875263557120x y    + 17455451270400x y  - 4363862817600x y
--R        +
--R                          9 4                   10 2                   11
--R           727310469600x y  - 72731046960x  y  + 3305956680x
--R    *
--R          27
--R         z
```

17

```
--R  +
--R                     24                    22                  2 20
--R      3385299640320y   - 20311797841920x y   + 55857444065280x y
--R     +
--R                     3 18                 4 16                 5 14
--R      - 93095740108800x y   + 104732707622400x y   - 83786166097920x y
--R     +
--R                    6 12                 7 10                8 8
--R      48875263557120x y   - 20946541524480x y   + 6545794226400x y
--R     +
--R                    9 6                 10 4                11 2              12
--R      - 1454620939200x y  + 218193140880x  y  - 19835740080x  y  + 826489170x
--R    *
--R       24
--R      z
--R  +
--R                     26                 24                  2 22
--R      - 1388840878080y   + 9027465707520x y   - 27082397122560x y
--R     +
--R                    3 20                 4 18                5 16
--R      49651061391360x y   - 62063826739200x y   + 55857444065280x y
--R     +
--R                     6 14                 7 12                8 10
--R      - 37238296043520x y   + 18619148021760x y   - 6982180508160x y
--R     +
--R                    9 8                 10 6                11 4
--R      1939494585600x y  - 387898917120x  y  + 52895306880x  y
--R     +
--R                   12 2             13
--R      - 4407942240x  y  + 169536240x
--R    *
--R       21
--R      z
--R  +
--R                   28                 26                  2 24
--R      462946959360y   - 3240628715520x y   + 10532043325440x y
--R     +
--R                     3 22                 4 20                 5 18
--R      - 21064086650880x y   + 28963119144960x y   - 28963119144960x y
--R     +
--R                    6 16                 7 14                8 12
--R      21722339358720x y   - 12412765347840x y   + 5430584839680x y
--R     +
--R                     9 10                10 8                11 6
--R      - 1810194946560x y   + 452548736640x  y  - 82281588480x  y
--R     +
--R                   12 4             13 2            14
--R      10285198560x  y  - 791169120x  y  + 28256040x
--R    *
--R       18
```

```
--R      z
--R  +
--R                      30                28                 2 26
--R      - 123452522496y   + 925893918720x y   - 3240628715520x y
--R      +
--R                  3 24              4 22                5 20
--R      7021362216960x y   - 10532043325440x y   + 11585247657984x y
--R      +
--R                    6 18             7 16               8 14
--R      - 9654373048320x y   + 6206382673920x y   - 3103191336960x y
--R      +
--R                  9 12             10 10              11 8
--R      1206796631040x y   - 362038989312x y   + 82281588480x  y
--R      +
--R                   12 6            13 4             14 2            15
--R      - 13713598080x y + 1582338240x y - 113024160x  y + 3767472x
--R    *
--R       15
--R      z
--R  +
--R                  32              30              2 28
--R      25719275520y   - 205754204160x y   + 771578265600x y
--R      +
--R                  3 26              4 24               5 22
--R      - 1800349286400x y + 2925567590400x y   - 3510681108480x y
--R      +
--R                 6 20              7 18               8 16
--R      3218124349440x y   - 2298660249600x y   + 1292996390400x y
--R      +
--R                  9 14             10 12             11 10
--R      - 574665062400x y   + 201132771840x y   - 54854392320x  y
--R      +
--R                  12 8           13 6            14 4            15 2
--R      11427998400x  y - 1758153600x  y + 188373600x  y - 12558240x  y
--R      +
--R              16
--R      392445x
--R    *
--R       12
--R      z
--R  +
--R                  34              32              2 30
--R      - 4034396160y   + 34292367360x y   - 137169469440x y
--R      +
--R                 3 28              4 26              5 24
--R      342923673600x y   - 600116428800x y   + 780151357440x y
--R      +
--R                  6 22             7 20              8 18
--R      - 780151357440x y   + 612976066560x y   - 383110041600x y
--R      +
```

19

$$
\begin{aligned}
--R \quad &\big(191555020800\,x^{9}y^{16} - 76622008320\,x^{10}y^{14} + 24379729920\,x^{11}y^{12} \\
--R \quad &+\; -\,6094932480\,x^{12}y^{10} + 1172102400\,x^{13}y^{8} - 167443200\,x^{14}y^{6} + 16744320\,x^{15}y^{4} \\
--R \quad &+\; -\,1046520\,x^{16}y^{2} + 30780\,x^{17}\big) \\
--R \quad *\; z^{9} \\
--R \quad +\; \big(448266240\,y^{36} - 4034396160\,x\,y^{34} + 17146183680\,x^{2}y^{32} - 45723156480\,x^{3}y^{30} \\
--R \quad &+\; 85730918400\,x^{4}y^{28} - 120023285760\,x^{5}y^{26} + 130025226240\,x^{6}y^{24} \\
--R \quad &+\; -\,111450193920\,x^{7}y^{22} + 76622008320\,x^{8}y^{20} - 42567782400\,x^{9}y^{18} \\
--R \quad &+\; 19155502080\,x^{10}y^{16} - 6965637120\,x^{11}y^{14} + 2031644160\,x^{12}y^{12} \\
--R \quad &+\; -\,468840960\,x^{13}y^{10} + 83721600\,x^{14}y^{8} - 11162880\,x^{15}y^{6} + 1046520\,x^{16}y^{4} \\
--R \quad &+\; -\,61560\,x^{17}y^{2} + 1710\,x^{18}\big) \\
--R \quad *\; z^{6} \\
--R \quad +\; \big(-\,31457280\,y^{38} + 298844160\,x\,y^{36} - 1344798720\,x^{2}y^{34} + 3810263040\,x^{3}y^{32} \\
--R \quad &+\; -\,7620526080\,x^{4}y^{30} + 11430789120\,x^{5}y^{28} - 13335920640\,x^{6}y^{26} \\
--R \quad &+\; 12383354880\,x^{7}y^{24} - 9287516160\,x^{8}y^{22} + 5675704320\,x^{9}y^{20} - 2837852160\,x^{10}y^{18} \\
--R \quad &+\; 1160939520\,x^{11}y^{16} - 386979840\,x^{12}y^{14} + 104186880\,x^{13}y^{12} - 22325760\,x^{14}y^{10} \\
--R \quad &+\; 3720960\,x^{15}y^{8} - 465120\,x^{16}y^{6} + 41040\,x^{17}y^{4} - 2280\,x^{18}y^{2} + 60\,x^{19}\big) \\
--R \quad *\; z^{3}
\end{aligned}
$$

```
--R   +
--R             40              38              2 36              3 34
--R    1048576y  - 10485760x y   + 49807360x y   - 149422080x y
--R   +
--R              4 32            5 30            6 28            7 26
--R    317521920x y  - 508035072x y   + 635043840x y   - 635043840x y
--R   +
--R              8 24            9 22            10 20           11 18
--R    515973120x y  - 343982080x y   + 189190144x   y   - 85995520x  y
--R   +
--R              12 16           13 14           14 12           15 10          16 8
--R    32248320x  y  - 9922560x  y  + 2480640x  y   - 496128x  y   + 77520x  y
--R   +
--R           17 6       18 4      19 2     20
--R    - 9120x  y  + 760x  y  - 40x  y  + x
--R                                              Type: Polynomial(Integer)
--E 45


--S 46 of 63
factor(%)
--R
--R
--R            3     2     20
--R   (46)  (3z  - 2y  + x)
--R                                          Type: Factored(Polynomial(Integer))
--E 46


--S 47 of 63
expand((sin(x) - 2*cos(y)**2 + 3*tan(z)**3)**20)
--R
--R
--R   (47)
--R                    60                                           2       57
--R    3486784401tan(z)   + (23245229340sin(x) - 46490458680cos(y) )tan(z)
--R   +
--R                    2                   2                            4
--R      (73609892910sin(x)  - 294439571640cos(y) sin(x) + 294439571640cos(y) )
--R    *
--R           54
--R      tan(z)
--R   +
--R                    3                   2     2
--R       147219785820sin(x)  - 883318714920cos(y) sin(x)
--R      +
--R                    4                   6
--R       1766637429840cos(y) sin(x) - 1177758286560cos(y)
--R    *
--R           51
--R      tan(z)
--R   +
```

```
--R                              4                          2        3
--R          208561363245sin(x)  - 1668490905960cos(y) sin(x)
--R       +
--R                              4       2                        6
--R          5005472717880cos(y) sin(x)  - 6673963623840cos(y) sin(x)
--R       +
--R                            8
--R          3336981811920cos(y)
--R     *
--R             48
--R        tan(z)
--R     +
--R                              5                          2        4
--R          222465454128sin(x)  - 2224654541280cos(y) sin(x)
--R       +
--R                              4       3                        6       2
--R          8898618165120cos(y) sin(x)  - 17797236330240cos(y) sin(x)
--R       +
--R                             8                           10
--R          17797236330240cos(y) sin(x) - 7118894532096cos(y)
--R     *
--R             45
--R        tan(z)
--R     +
--R                              6                          2        5
--R          185387878440sin(x)  - 2224654541280cos(y) sin(x)
--R       +
--R                              4       4                         6       3
--R          11123272706400cos(y) sin(x)  - 29662060550400cos(y) sin(x)
--R       +
--R                              8       2                         10
--R          44493090825600cos(y) sin(x)  - 35594472660480cos(y)  sin(x)
--R       +
--R                             12
--R          11864824220160cos(y)
--R     *
--R             42
--R        tan(z)
--R     +
--R                              7                          2        6
--R          123591918960sin(x)  - 1730286865440cos(y) sin(x)
--R       +
--R                              4       5                         6       4
--R          10381721192640cos(y) sin(x)  - 34605737308800cos(y) sin(x)
--R       +
--R                              8       3                         10      2
--R          69211474617600cos(y) sin(x)  - 83053769541120cos(y)  sin(x)
--R       +
--R                              12                          14
--R          55369179694080cos(y)  sin(x) - 15819765626880cos(y)
```

```
--R    *
--R            39
--R        tan(z)
--R    +
--R                              8                        2        7
--R          66945622770sin(x)  - 1071129964320cos(y) sin(x)
--R        +
--R                                4       6                        6       5
--R          7497909750240cos(y) sin(x)  - 29991639000960cos(y) sin(x)
--R        +
--R                                 8       4                         10       3
--R          74979097502400cos(y) sin(x)  - 119966556003840cos(y)  sin(x)
--R        +
--R                                  12       2                        14
--R          119966556003840cos(y)  sin(x)  - 68552317716480cos(y)  sin(x)
--R        +
--R                                 16
--R          17138079429120cos(y)
--R    *
--R            36
--R        tan(z)
--R    +
--R                              9                       2        8
--R          29753610120sin(x)  - 535564982160cos(y) sin(x)
--R        +
--R                                4       7                        6       6
--R          4284519857280cos(y) sin(x)  - 19994426000640cos(y) sin(x)
--R        +
--R                                 8       5                         10       4
--R          59983278001920cos(y) sin(x)  - 119966556003840cos(y)  sin(x)
--R        +
--R                                  12       3                         14       2
--R          159955408005120cos(y)  sin(x)  - 137104635432960cos(y)  sin(x)
--R        +
--R                                 16                         18
--R          68552317716480cos(y)  sin(x) - 15233848381440cos(y)
--R    *
--R            33
--R        tan(z)
--R    +
--R                              10                       2        9
--R          10909657044sin(x)   - 218193140880cos(y) sin(x)
--R        +
--R                                4       8                        6       7
--R          1963738267920cos(y) sin(x)  - 10473270762240cos(y) sin(x)
--R        +
--R                                 8       6                        10       5
--R          36656447667840cos(y) sin(x)  - 87975474402816cos(y)  sin(x)
--R        +
--R                                  12       4                        14       3
```

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```
--R        146625790671360cos(y)  sin(x)  - 167572332195840cos(y)  sin(x)
--R      +
--R                             16     2                           18
--R        125679249146880cos(y)  sin(x)  - 55857444065280cos(y)  sin(x)
--R      +
--R                             20
--R        11171488813056cos(y)
--R    *
--R          30
--R       tan(z)
--R    +
--R                          11                            2     10
--R        3305956680sin(x)    - 72731046960cos(y) sin(x)
--R      +
--R                            4     9                          6     8
--R        727310469600cos(y) sin(x)  - 4363862817600cos(y) sin(x)
--R      +
--R                             8     7                           10     6
--R        17455451270400cos(y) sin(x)  - 48875263557120cos(y)  sin(x)
--R      +
--R                             12     5                            14     4
--R        97750527114240cos(y)  sin(x)  - 139643610163200cos(y)  sin(x)
--R      +
--R                              16     3                            18     2
--R        139643610163200cos(y)  sin(x)  - 93095740108800cos(y)  sin(x)
--R      +
--R                             20                           22
--R        37238296043520cos(y)  sin(x) - 6770599280640cos(y)
--R    *
--R          27
--R       tan(z)
--R    +
--R                         12                          2     11
--R        826489170sin(x)    - 19835740080cos(y) sin(x)
--R      +
--R                           4     10                           6     9
--R        218193140880cos(y) sin(x)    - 1454620939200cos(y) sin(x)
--R      +
--R                            8     8                            10     7
--R        6545794226400cos(y) sin(x)  - 20946541524480cos(y)  sin(x)
--R      +
--R                             12     6                           14     5
--R        48875263557120cos(y)  sin(x)  - 83786166097920cos(y)  sin(x)
--R      +
--R                              16     4                           18     3
--R        104732707622400cos(y)  sin(x)  - 93095740108800cos(y)  sin(x)
--R      +
--R                            20     2                           22
--R        55857444065280cos(y)  sin(x)  - 20311797841920cos(y)  sin(x)
--R      +
```

```
--R                           24
--R         3385299640320cos(y)
--R    *
--R            24
--R       tan(z)
--R   +
--R                      13                      2      12
--R         169536240sin(x)    - 4407942240cos(y) sin(x)
--R       +
--R                        4      11                        6      10
--R         52895306880cos(y) sin(x)    - 387898917120cos(y) sin(x)
--R       +
--R                          8      9                        10      8
--R         1939494585600cos(y) sin(x) - 6982180508160cos(y)  sin(x)
--R       +
--R                           12      7                        14      6
--R         18619148021760cos(y)  sin(x)  - 37238296043520cos(y)  sin(x)
--R       +
--R                           16      5                        18      4
--R         55857444065280cos(y)  sin(x)  - 62063826739200cos(y)  sin(x)
--R       +
--R                           20      3                        22      2
--R         49651061391360cos(y)  sin(x)  - 27082397122560cos(y)  sin(x)
--R       +
--R                          24                        26
--R         9027465707520cos(y)  sin(x) - 1388840878080cos(y)
--R    *
--R            21
--R       tan(z)
--R   +
--R                     14                      2      13
--R         28256040sin(x)    - 791169120cos(y) sin(x)
--R       +
--R                         4      12                       6      11
--R         10285198560cos(y) sin(x)    - 82281588480cos(y) sin(x)
--R       +
--R                        8      10                        10      9
--R         452548736640cos(y) sin(x)    - 1810194946560cos(y)  sin(x)
--R       +
--R                          12      8                        14      7
--R         5430584839680cos(y)  sin(x)  - 12412765347840cos(y)  sin(x)
--R       +
--R                           16      6                        18      5
--R         21722339358720cos(y)  sin(x)  - 28963119144960cos(y)  sin(x)
--R       +
--R                           20      4                        22      3
--R         28963119144960cos(y)  sin(x)  - 21064086650880cos(y)  sin(x)
--R       +
--R                           24      2                        26
--R         10532043325440cos(y)  sin(x)  - 3240628715520cos(y)  sin(x)
```

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```
--R        +
--R                          28
--R           462946959360cos(y)
--R     *
--R             18
--R        tan(z)
--R     +
--R                        15                    2     14                        4      13
--R           3767472sin(x)   - 113024160cos(y) sin(x)   + 1582338240cos(y) sin(x)
--R        +
--R                          6     12                    8      11
--R           - 13713598080cos(y) sin(x)   + 82281588480cos(y) sin(x)
--R        +
--R                           10      10                      12      9
--R           - 362038989312cos(y)  sin(x)   + 1206796631040cos(y)  sin(x)
--R        +
--R                            14      8                      16      7
--R           - 3103191336960cos(y)  sin(x)  + 6206382673920cos(y)  sin(x)
--R        +
--R                            18      6                       20      5
--R           - 9654373048320cos(y)  sin(x)  + 11585247657984cos(y)  sin(x)
--R        +
--R                             22      4                      24      3
--R           - 10532043325440cos(y)  sin(x)  + 7021362216960cos(y)  sin(x)
--R        +
--R                            26      2                      28
--R           - 3240628715520cos(y)  sin(x)  + 925893918720cos(y)  sin(x)
--R        +
--R                            30
--R           - 123452522496cos(y)
--R     *
--R             15
--R        tan(z)
--R     +
--R                        16                    2     15                      4      14
--R           392445sin(x)   - 12558240cos(y) sin(x)   + 188373600cos(y) sin(x)
--R        +
--R                          6      13                    8      12
--R           - 1758153600cos(y) sin(x)   + 11427998400cos(y) sin(x)
--R        +
--R                          10      11                    12      10
--R           - 54854392320cos(y)  sin(x)   + 201132771840cos(y)  sin(x)
--R        +
--R                          14      9                       16      8
--R           - 574665062400cos(y)  sin(x)   + 1292996390400cos(y)  sin(x)
--R        +
--R                           18      7                      20      6
--R           - 2298660249600cos(y)  sin(x)  + 3218124349440cos(y)  sin(x)
--R        +
--R                            22      5                       24      4
```

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```
--R          - 3510681108480cos(y)  sin(x)  + 2925567590400cos(y)  sin(x)
--R       +
--R                             26       3                        28       2
--R          - 1800349286400cos(y)  sin(x)  + 771578265600cos(y)  sin(x)
--R       +
--R                             30                        32
--R          - 205754204160cos(y)  sin(x) + 25719275520cos(y)
--R     *
--R            12
--R       tan(z)
--R     +
--R                      17                2      16                    4      15
--R          30780sin(x)   - 1046520cos(y) sin(x)   + 16744320cos(y) sin(x)
--R       +
--R                         6      14                8      13
--R          - 167443200cos(y) sin(x)   + 1172102400cos(y) sin(x)
--R       +
--R                          10      12                 12      11
--R          - 6094932480cos(y)  sin(x)   + 24379729920cos(y)  sin(x)
--R       +
--R                           14      10                  16      9
--R          - 76622008320cos(y)  sin(x)   + 191555020800cos(y)  sin(x)
--R       +
--R                            18      8                   20      7
--R          - 383110041600cos(y)  sin(x)  + 612976066560cos(y)  sin(x)
--R       +
--R                            22      6                   24      5
--R          - 780151357440cos(y)  sin(x)  + 780151357440cos(y)  sin(x)
--R       +
--R                            26      4                   28      3
--R          - 600116428800cos(y)  sin(x)  + 342923673600cos(y)  sin(x)
--R       +
--R                            30      2                   32
--R          - 137169469440cos(y)  sin(x)  + 34292367360cos(y)  sin(x)
--R       +
--R                           34
--R          - 4034396160cos(y)
--R     *
--R            9
--R       tan(z)
--R     +
--R                     18               2      17                  4      16
--R          1710sin(x)   - 61560cos(y) sin(x)   + 1046520cos(y) sin(x)
--R       +
--R                        6      15                8      14
--R          - 11162880cos(y) sin(x)   + 83721600cos(y) sin(x)
--R       +
--R                         10      13                 12      12
--R          - 468840960cos(y)  sin(x)   + 2031644160cos(y)  sin(x)
--R       +
```

```
--R                            14      11                      16      10
--R        - 6965637120cos(y)  sin(x)   + 19155502080cos(y)  sin(x)
--R      +
--R                            18     9                      20     8
--R        - 42567782400cos(y)  sin(x) + 76622008320cos(y)  sin(x)
--R      +
--R                             22     7                      24     6
--R        - 111450193920cos(y)  sin(x)  + 130025226240cos(y)  sin(x)
--R      +
--R                             26     5                     28     4
--R        - 120023285760cos(y)  sin(x)  + 85730918400cos(y)  sin(x)
--R      +
--R                            30     3                     32     2
--R        - 45723156480cos(y)  sin(x)  + 17146183680cos(y)  sin(x)
--R      +
--R                           34                      36
--R        - 4034396160cos(y)  sin(x) + 448266240cos(y)
--R    *
--R           6
--R      tan(z)
--R  +
--R             19              2     18             4     17
--R        60sin(x)   - 2280cos(y) sin(x)   + 41040cos(y) sin(x)
--R      +
--R                    6     16               8     15
--R        - 465120cos(y) sin(x)   + 3720960cos(y) sin(x)
--R      +
--R                      10     14                  12     13
--R        - 22325760cos(y)  sin(x)   + 104186880cos(y)  sin(x)
--R      +
--R                       14     12                   16     11
--R        - 386979840cos(y)  sin(x)   + 1160939520cos(y)  sin(x)
--R      +
--R                        18     10                   20     9
--R        - 2837852160cos(y)  sin(x)   + 5675704320cos(y)  sin(x)
--R      +
--R                        22     8                    24     7
--R        - 9287516160cos(y)  sin(x)  + 12383354880cos(y)  sin(x)
--R      +
--R                         26     6                    28     5
--R        - 13335920640cos(y)  sin(x)  + 11430789120cos(y)  sin(x)
--R      +
--R                        30     4                    32     3
--R        - 7620526080cos(y)  sin(x)  + 3810263040cos(y)  sin(x)
--R      +
--R                         34     2                    36                      38
--R        - 1344798720cos(y)  sin(x)  + 298844160cos(y)  sin(x) - 31457280cos(y)
--R    *
--R           3
--R      tan(z)
```

```
--R   +
--R          20              2     19           4      18             6      17
--R    sin(x)   - 40cos(y) sin(x)   + 760cos(y) sin(x)   - 9120cos(y) sin(x)
--R   +
--R             8     16              10     15               12     14
--R    77520cos(y) sin(x)   - 496128cos(y)  sin(x)   + 2480640cos(y)  sin(x)
--R   +
--R                 14     13                16     12
--R    - 9922560cos(y)  sin(x)   + 32248320cos(y)  sin(x)
--R   +
--R                  18     11                 20     10
--R    - 85995520cos(y)  sin(x)   + 189190144cos(y)  sin(x)
--R   +
--R                   22     9                 24     8
--R    - 343982080cos(y)  sin(x)  + 515973120cos(y)  sin(x)
--R   +
--R                   26     7                 28     6
--R    - 635043840cos(y)  sin(x)  + 635043840cos(y)  sin(x)
--R   +
--R                   30     5                 32     4
--R    - 508035072cos(y)  sin(x)  + 317521920cos(y)  sin(x)
--R   +
--R                   34     3                 36     2
--R    - 149422080cos(y)  sin(x)  + 49807360cos(y)  sin(x)
--R   +
--R                   38                      40
--R    - 10485760cos(y)  sin(x) + 1048576cos(y)
--R                                            Type: Expression(Integer)
--E 47

--S 48 of 63
factor(%)
--R
--R
--R   (48)
--R                   60                                      2     57
--R    3486784401tan(z)   + (23245229340sin(x) - 46490458680cos(y) )tan(z)
--R   +
--R                   2                  2                           4
--R      (73609892910sin(x)  - 294439571640cos(y) sin(x) + 294439571640cos(y) )
--R    *
--R           54
--R      tan(z)
--R   +
--R                         3                  2     2
--R         147219785820sin(x)  - 883318714920cos(y) sin(x)
--R       +
--R                         4                 6
--R         1766637429840cos(y) sin(x) - 1177758286560cos(y)
--R    *
```

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```
--R             51
--R      tan(z)
--R   +
--R                              4                            2      3
--R         208561363245sin(x)  - 1668490905960cos(y) sin(x)
--R      +
--R                              4      2                          6
--R         5005472717880cos(y) sin(x)  - 6673963623840cos(y) sin(x)
--R      +
--R                              8
--R         3336981811920cos(y)
--R    *
--R             48
--R      tan(z)
--R   +
--R                              5                            2      4
--R         222465454128sin(x)  - 2224654541280cos(y) sin(x)
--R      +
--R                              4      3                          6      2
--R         8898618165120cos(y) sin(x)  - 17797236330240cos(y) sin(x)
--R      +
--R                              8                            10
--R         17797236330240cos(y) sin(x) - 7118894532096cos(y)
--R    *
--R             45
--R      tan(z)
--R   +
--R                              6                            2      5
--R         185387878440sin(x)  - 2224654541280cos(y) sin(x)
--R      +
--R                              4      4                          6      3
--R         11123272706400cos(y) sin(x)  - 29662060550400cos(y) sin(x)
--R      +
--R                              8      2                          10
--R         44493090825600cos(y) sin(x)  - 35594472660480cos(y)  sin(x)
--R      +
--R                              12
--R         11864824220160cos(y)
--R    *
--R             42
--R      tan(z)
--R   +
--R                              7                            2      6
--R         123591918960sin(x)  - 1730286865440cos(y) sin(x)
--R      +
--R                              4      5                          6      4
--R         10381721192640cos(y) sin(x)  - 34605737308800cos(y) sin(x)
--R      +
--R                              8      3                          10      2
--R         69211474617600cos(y) sin(x)  - 83053769541120cos(y)  sin(x)
```

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```
--R        +
--R                                 12                            14
--R           55369179694080cos(y)  sin(x) - 15819765626880cos(y)
--R     *
--R             39
--R         tan(z)
--R     +
--R                               8                        2       7
--R           66945622770sin(x)  - 1071129964320cos(y) sin(x)
--R         +
--R                               4       6                        6       5
--R           7497909750240cos(y) sin(x)  - 29991639000960cos(y) sin(x)
--R         +
--R                               8       4                        10        3
--R           74979097502400cos(y) sin(x)  - 119966556003840cos(y)  sin(x)
--R         +
--R                                12       2                        14
--R           119966556003840cos(y)  sin(x)  - 68552317716480cos(y)  sin(x)
--R         +
--R                               16
--R           17138079429120cos(y)
--R     *
--R             36
--R         tan(z)
--R     +
--R                               9                        2       8
--R           29753610120sin(x)  - 535564982160cos(y) sin(x)
--R         +
--R                               4       7                        6       6
--R           4284519857280cos(y) sin(x)  - 19994426000640cos(y) sin(x)
--R         +
--R                               8       5                        10        4
--R           59983278001920cos(y) sin(x)  - 119966556003840cos(y)  sin(x)
--R         +
--R                                12       3                        14        2
--R           159955408005120cos(y)  sin(x)  - 137104635432960cos(y)  sin(x)
--R         +
--R                               16                        18
--R           68552317716480cos(y)  sin(x) - 15233848381440cos(y)
--R     *
--R             33
--R         tan(z)
--R     +
--R                               10                       2       9
--R           10909657044sin(x)   - 218193140880cos(y) sin(x)
--R         +
--R                               4       8                        6       7
--R           1963738267920cos(y) sin(x)  - 10473270762240cos(y) sin(x)
--R         +
--R                                8       6                        10        5
```

```
--R          36656447667840cos(y) sin(x)  - 87975474402816cos(y)  sin(x)
--R       +
--R                           12      4                            14      3
--R          146625790671360cos(y)  sin(x)  - 167572332195840cos(y)  sin(x)
--R       +
--R                           16      2                          18
--R          125679249146880cos(y)  sin(x)  - 55857444065280cos(y)  sin(x)
--R       +
--R                           20
--R          11171488813056cos(y)
--R     *
--R            30
--R        tan(z)
--R     +
--R                         11                         2      10
--R          3305956680sin(x)   - 72731046960cos(y) sin(x)
--R       +
--R                          4      9                          6      8
--R          727310469600cos(y) sin(x)  - 4363862817600cos(y) sin(x)
--R       +
--R                           8      7                           10      6
--R          17455451270400cos(y) sin(x)  - 48875263557120cos(y)  sin(x)
--R       +
--R                           12      5                            14      4
--R          97750527114240cos(y)  sin(x)  - 139643610163200cos(y)  sin(x)
--R       +
--R                            16      3                          18      2
--R          139643610163200cos(y)  sin(x)  - 93095740108800cos(y)  sin(x)
--R       +
--R                          20                          22
--R          37238296043520cos(y)  sin(x) - 6770599280640cos(y)
--R     *
--R            27
--R        tan(z)
--R     +
--R                        12                      2      11
--R          826489170sin(x)   - 19835740080cos(y) sin(x)
--R       +
--R                          4      10                         6      9
--R          218193140880cos(y) sin(x)   - 1454620939200cos(y) sin(x)
--R       +
--R                          8      8                           10      7
--R          6545794226400cos(y) sin(x)  - 20946541524480cos(y)  sin(x)
--R       +
--R                           12      6                            14      5
--R          48875263557120cos(y)  sin(x)  - 83786166097920cos(y)  sin(x)
--R       +
--R                            16      4                          18      3
--R          104732707622400cos(y)  sin(x)  - 93095740108800cos(y)  sin(x)
--R       +
```

```
--R                               20     2                              22
--R          55857444065280cos(y)  sin(x)  - 20311797841920cos(y)  sin(x)
--R      +
--R                             24
--R          3385299640320cos(y)
--R    *
--R            24
--R        tan(z)
--R    +
--R                             13                          2    12
--R          169536240sin(x)   - 4407942240cos(y) sin(x)
--R      +
--R                            4    11                          6    10
--R          52895306880cos(y) sin(x)   - 387898917120cos(y) sin(x)
--R      +
--R                             8    9                           10    8
--R          1939494585600cos(y) sin(x)  - 6982180508160cos(y)  sin(x)
--R      +
--R                              12    7                          14    6
--R          18619148021760cos(y)  sin(x)  - 37238296043520cos(y)  sin(x)
--R      +
--R                              16    5                          18    4
--R          55857444065280cos(y)  sin(x)  - 62063826739200cos(y)  sin(x)
--R      +
--R                              20    3                          22    2
--R          49651061391360cos(y)  sin(x)  - 27082397122560cos(y)  sin(x)
--R      +
--R                             24                          26
--R          9027465707520cos(y)  sin(x) - 1388840878080cos(y)
--R    *
--R            21
--R        tan(z)
--R    +
--R                            14                          2    13
--R          28256040sin(x)   - 791169120cos(y) sin(x)
--R      +
--R                            4    12                         6    11
--R          10285198560cos(y) sin(x)   - 82281588480cos(y) sin(x)
--R      +
--R                            8    10                          10    9
--R          452548736640cos(y) sin(x)   - 1810194946560cos(y)  sin(x)
--R      +
--R                             12    8                         14    7
--R          5430584839680cos(y)  sin(x) - 12412765347840cos(y)  sin(x)
--R      +
--R                              16    6                         18    5
--R          21722339358720cos(y)  sin(x) - 28963119144960cos(y)  sin(x)
--R      +
--R                              20    4                         22    3
--R          28963119144960cos(y)  sin(x) - 21064086650880cos(y)  sin(x)
```

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```
--R       +
--R                              24      2                           26
--R          10532043325440cos(y)  sin(x)  - 3240628715520cos(y)  sin(x)
--R       +
--R                            28
--R          462946959360cos(y)
--R    *
--R           18
--R       tan(z)
--R    +
--R                      15                     2      14                      4      13
--R          3767472sin(x)  - 113024160cos(y) sin(x)  + 1582338240cos(y) sin(x)
--R       +
--R                            6      12                     8      11
--R        - 13713598080cos(y) sin(x)  + 82281588480cos(y) sin(x)
--R       +
--R                            10      10                          12      9
--R        - 362038989312cos(y)  sin(x)  + 1206796631040cos(y)  sin(x)
--R       +
--R                             14      8                       16      7
--R        - 3103191336960cos(y)  sin(x)  + 6206382673920cos(y)  sin(x)
--R       +
--R                             18      6                        20      5
--R        - 9654373048320cos(y)  sin(x)  + 11585247657984cos(y)  sin(x)
--R       +
--R                              22      4                      24      3
--R        - 10532043325440cos(y)  sin(x)  + 7021362216960cos(y)  sin(x)
--R       +
--R                             26      2                      28
--R        - 3240628715520cos(y)  sin(x)  + 925893918720cos(y)  sin(x)
--R       +
--R                            30
--R        - 123452522496cos(y)
--R    *
--R           15
--R       tan(z)
--R    +
--R                      16                     2      15                     4      14
--R          392445sin(x)  - 12558240cos(y) sin(x)   + 188373600cos(y) sin(x)
--R       +
--R                          6      13                     8      12
--R        - 1758153600cos(y) sin(x)   + 11427998400cos(y) sin(x)
--R       +
--R                           10      11                        12      10
--R        - 54854392320cos(y)  sin(x)  + 201132771840cos(y)  sin(x)
--R       +
--R                            14      9                        16      8
--R        - 574665062400cos(y)  sin(x)  + 1292996390400cos(y)  sin(x)
--R       +
--R                            18      7                        20      6
```

```
--R            - 2298660249600cos(y)  sin(x)   + 3218124349440cos(y)  sin(x)
--R        +
--R                               22      5                            24      4
--R            - 3510681108480cos(y)  sin(x)   + 2925567590400cos(y)  sin(x)
--R        +
--R                               26      3                            28      2
--R            - 1800349286400cos(y)  sin(x)   + 771578265600cos(y)  sin(x)
--R        +
--R                               30                            32
--R            - 205754204160cos(y)  sin(x) + 25719275520cos(y)
--R      *
--R              12
--R          tan(z)
--R      +
--R                        17                  2      16                        4      15
--R            30780sin(x)    - 1046520cos(y) sin(x)    + 16744320cos(y) sin(x)
--R        +
--R                               6      14                    8      13
--R            - 167443200cos(y) sin(x)    + 1172102400cos(y) sin(x)
--R        +
--R                               10      12                     12      11
--R            - 6094932480cos(y)  sin(x)    + 24379729920cos(y)  sin(x)
--R        +
--R                               14      10                            16      9
--R            - 76622008320cos(y)  sin(x)    + 191555020800cos(y)  sin(x)
--R        +
--R                               18      8                            20      7
--R            - 383110041600cos(y)  sin(x)   + 612976066560cos(y)  sin(x)
--R        +
--R                               22      6                            24      5
--R            - 780151357440cos(y)  sin(x)   + 780151357440cos(y)  sin(x)
--R        +
--R                               26      4                            28      3
--R            - 600116428800cos(y)  sin(x)   + 342923673600cos(y)  sin(x)
--R        +
--R                               30      2                            32
--R            - 137169469440cos(y)  sin(x)   + 34292367360cos(y)  sin(x)
--R        +
--R                               34
--R            - 4034396160cos(y)
--R      *
--R             9
--R          tan(z)
--R      +
--R                       18                  2      17                      4      16
--R            1710sin(x)    - 61560cos(y) sin(x)    + 1046520cos(y) sin(x)
--R        +
--R                               6      15                    8      14
--R            - 11162880cos(y) sin(x)    + 83721600cos(y) sin(x)
--R        +
```

```
--R                             10   13                        12   12
--R        - 468840960cos(y)  sin(x)   + 2031644160cos(y)  sin(x)
--R      +
--R                             14   11                        16   10
--R        - 6965637120cos(y)  sin(x)   + 19155502080cos(y)  sin(x)
--R      +
--R                             18  9                        20   8
--R        - 42567782400cos(y)  sin(x)  + 76622008320cos(y)  sin(x)
--R      +
--R                              22  7                        24   6
--R        - 111450193920cos(y)  sin(x)  + 130025226240cos(y)  sin(x)
--R      +
--R                              26  5                        28   4
--R        - 120023285760cos(y)  sin(x)  + 85730918400cos(y)  sin(x)
--R      +
--R                             30  3                        32   2
--R        - 45723156480cos(y)  sin(x)  + 17146183680cos(y)  sin(x)
--R      +
--R                            34                        36
--R        - 4034396160cos(y)  sin(x) + 448266240cos(y)
--R    *
--R          6
--R      tan(z)
--R    +
--R              19           2   18              4   17
--R        60sin(x)  - 2280cos(y) sin(x)  + 41040cos(y) sin(x)
--R      +
--R                   6   16              8   15
--R        - 465120cos(y) sin(x)  + 3720960cos(y) sin(x)
--R      +
--R                     10   14                       12   13
--R        - 22325760cos(y)  sin(x)  + 104186880cos(y)  sin(x)
--R      +
--R                      14   12                        16   11
--R        - 386979840cos(y)  sin(x)  + 1160939520cos(y)  sin(x)
--R      +
--R                       18   10                       20  9
--R        - 2837852160cos(y)  sin(x)  + 5675704320cos(y)  sin(x)
--R      +
--R                       22  8                        24  7
--R        - 9287516160cos(y)  sin(x)  + 12383354880cos(y)  sin(x)
--R      +
--R                        26  6                        28  5
--R        - 13335920640cos(y)  sin(x)  + 11430789120cos(y)  sin(x)
--R      +
--R                       30  4                        32  3
--R        - 7620526080cos(y)  sin(x)  + 3810263040cos(y)  sin(x)
--R      +
--R                        34  2                       36                       38
--R        - 1344798720cos(y)  sin(x)  + 298844160cos(y)  sin(x) - 31457280cos(y)
```

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```
--R    *
--R            3
--R       tan(z)
--R   +
--R          20          2    19            4    18           6    17
--R     sin(x)  - 40cos(y) sin(x)  + 760cos(y) sin(x)  - 9120cos(y) sin(x)
--R   +
--R              8    16              10    15              12    14
--R     77520cos(y) sin(x)  - 496128cos(y)  sin(x)  + 2480640cos(y)  sin(x)
--R   +
--R                14    13              16    12
--R     - 9922560cos(y)  sin(x)  + 32248320cos(y)  sin(x)
--R   +
--R                 18    11               20    10
--R     - 85995520cos(y)  sin(x)  + 189190144cos(y)  sin(x)
--R   +
--R                  22    9                24    8
--R     - 343982080cos(y)  sin(x) + 515973120cos(y)  sin(x)
--R   +
--R                  26    7                28    6
--R     - 635043840cos(y)  sin(x) + 635043840cos(y)  sin(x)
--R   +
--R                  30    5                32    4
--R     - 508035072cos(y)  sin(x) + 317521920cos(y)  sin(x)
--R   +
--R                  34    3               36    2
--R     - 149422080cos(y)  sin(x) + 49807360cos(y)  sin(x)
--R   +
--R                  38                  40
--R     - 10485760cos(y)  sin(x) + 1048576cos(y)
--R                                   Type: Factored(Expression(Integer))
--E 48
```

$$\overline{\text{expand}[(1 - c^2)^5(1 - s^2)^5(c^2 + s^2)^{10}] => c^{10}s^{10}}$$
    when $c^2 + s^2 = 1$ [modification of a problem due to Richard Liska]
$$\text{— * —}$$

```
--S 49 of 63
expand((1 - c**2)**5 * (1 - s**2)**5 * (c**2 + s**2)**10)
--R
--R
--R   (49)
--R       10     8      6      4     2       30
--R     (c   - 5c  + 10c  - 10c  + 5c  - 1)s
--R   +
--R         12      10       8       6       4      2      28
--R      (10c   - 55c   + 125c  - 150c  + 100c  - 35c  + 5)s
```

```
--R   +
--R          14        12        10         8        6        4       2       26
--R     (45c    - 275c    + 710c    - 1000c  + 825c  - 395c  + 100c  - 10)s
--R   +
--R           16        14        12        10        8        6        4       2
--R     120c    - 825c    + 2425c   - 3960c   + 3900c - 2345c + 825c - 150c
--R       +
--R         10
--R   *
--R        24
--R       s
--R   +
--R           18        16        14         12         10        8        6
--R     210c    - 1650c   + 5550c   - 10450c   + 12055c   - 8735c + 3900c
--R       +
--R             4       2
--R      - 1000c  + 125c  - 5
--R   *
--R        22
--R       s
--R   +
--R           20         18        16         14         12         10        8
--R     252c    - 2310c   + 8970c   - 19470c   + 26060c   - 22253c   + 12055c
--R       +
--R             6        4       2
--R      - 3960c  + 710c  - 55c  + 1
--R   *
--R        20
--R       s
--R   +
--R           22         20         18         16         14         12        10
--R     210c    - 2310c   + 10500c   - 26400c   + 40875c   - 40645c   + 26060c
--R       +
--R             8        6        4       2
--R      - 10450c  + 2425c  - 275c  + 10c
--R   *
--R        18
--R       s
--R   +
--R           24         22         20         18         16         14        12
--R     120c    - 1650c   + 8970c   - 26400c   + 47400c   - 54615c   + 40875c
--R       +
--R             10        8        6       4
--R      - 19470c   + 5550c  - 825c  + 45c
--R   *
--R        16
--R       s
--R   +
--R           26         24         22         20         18         16        14
--R      45c    - 825c    + 5550c   - 19470c   + 40875c   - 54615c   + 47400c
```

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```
--R        +
--R                    12          10          8          6
--R          - 26400c     + 8970c    - 1650c   + 120c
--R     *
--R          14
--R         s
--R     +
--R                 28         26         24         22         20         18          16
--R          10c    - 275c    + 2425c    - 10450c    + 26060c    - 40645c    + 40875c
--R        +
--R                    14          12          10         8
--R          - 26400c     + 10500c    - 2310c    + 210c
--R     *
--R          12
--R         s
--R     +
--R              30         28         26         24         22         20          18
--R           c    - 55c    + 710c    - 3960c    + 12055c    - 22253c    + 26060c
--R        +
--R                    16          14          12          10
--R          - 19470c     + 8970c    - 2310c    + 252c
--R     *
--R          10
--R         s
--R     +
--R                 30         28          26         24         22         20          18
--R          - 5c    + 125c    - 1000c    + 3900c    - 8735c    + 12055c    - 10450c
--R        +
--R                16          14          12
--R          5550c    - 1650c    + 210c
--R     *
--R          8
--R         s
--R     +
--R                 30         28         26         24         22         20          18
--R          10c    - 150c    + 825c    - 2345c    + 3900c    - 3960c    + 2425c
--R        +
--R                  16         14
--R          - 825c    + 120c
--R     *
--R          6
--R         s
--R     +
--R                 30         28         26         24         22         20         18       16  4
--R       (- 10c    + 100c    - 395c    + 825c    - 1000c    + 710c    - 275c    + 45c  )s
--R     +
--R            30         28          26         24         22         20         18 2    30          28
--R       (5c    - 35c    + 100c    - 150c    + 125c    - 55c    + 10c  )s   - c    + 5c
--R     +
--R              26         24         22         20
```

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```
--R       - 10c    + 10c    - 5c    + c
--R                                                        Type: Polynomial(Integer)
--E 49


--S 50 of 63
groebner([%, c**2 + s**2 - 1])
--R
--R
--R            2    2      20      18       16      14      12      10
--R   (50)  [s  + c  - 1,c    - 5c    + 10c    - 10c    + 5c    - c   ]
--R                                          Type: List(Polynomial(Integer))
--E 50


--S 51 of 63
map(factor, %)
--R
--R
--R            2    2            5 10        5
--R   (51)  [s  + c  - 1,(c - 1) c   (c + 1) ]
--R                                      Type: List(Factored(Polynomial(Integer)))
--E 51
```

$$=> (x + y)(x - y) \ mod \ 3$$
— * —

```
--S 52 of 63
factor(4*x**2 - 21*x*y + 20*y**2 :: Polynomial(PrimeField(3)))
--R
--R   There are 22 exposed and 18 unexposed library operations named **
--R      having 2 argument(s) but none was determined to be applicable.
--R      Use HyperDoc Browse, or issue
--R                              )display op **
--R      to learn more about the available operations. Perhaps
--R      package-calling the operation or using coercions on the arguments
--R      will allow you to apply the operation.
--R
--R   Cannot find a definition or applicable library operation named **
--R      with argument type(s)
--R                                Variable(y)
--R                         Polynomial(PrimeField(3))
--R
--R      Perhaps you should use "@" to indicate the required return type,
--R      or "$" to specify which version of the function you need.
--E 52
```

$$=> 1/4(x + y)(2x + y[-1 + isqrt(3)])(2x + y[-1 - isqrt(3)])$$
— * —

40

```
--S 53 of 63
factor(x**3 + y**3, [rootOf(isqrt3**2 + 3)])
--R
--R
--R              - isqrt3 - 1                  isqrt3 - 1
--R   (52)  (y + ------------ x)(y + x)(y + ---------- x)
--R                    2                          2
--R                                      Type: Factored(Polynomial(AlgebraicNumber))
--E 53
```

—————-

Partial fraction decomposition $=> 3/(x+2) - 2/(x+1) + 2/(x+1)^2$
— * —

```
--S 54 of 63
(x**2 + 2*x + 3)/(x**3 + 4*x**2 + 5*x + 2)
--R
--R
--R            2
--R           x  + 2x + 3
--R   (53)  ------------------
--R          3     2
--R         x  + 4x  + 5x + 2
--R                                      Type: Fraction(Polynomial(Integer))
--E 54
```

```
--S 55 of 63
fullPartialFraction( _
   % :: Fraction UnivariatePolynomial(x, Fraction Integer))
--R
--R
--R            2         2        3
--R   (54)   - ----- + -------- + -----
--R           x + 1         2    x + 2
--R                    (x + 1)
--RType: FullPartialFractionExpansion(Fraction(Integer),UnivariatePolynomial(x,Fraction(Integer)))
--E 55
```

—————-

Noncommutative algebra: note that $(ABC)^{(-1)} = C^{(-1)}B^{(-1)}A^{(-1)}$
    $=> ABCACB - C^{(-1)}B^{(-1)}CB$
— * —

```
--S 56 of 63
A : SquareMatrix(2, Integer)
--R
--R                                                        Type: Void
--E 56
```

```
--S 57 of 63
B : SquareMatrix(2, Integer)
--R
--R                                                              Type: Void
--E 57


--S 58 of 63
C : SquareMatrix(2, Integer)
--R
--R                                                              Type: Void
--E 58


--S 59 of 63
(A*B*C - (A*B*C)**(-1)) * A*C*B
--R
--R
--R   A is declared as being in SquareMatrix(2,Integer) but has not been
--R      given a value.
--E 59
```

———-

Jacobi's identity: $[A, B, C] + [B, C, A] + [C, A, B] = 0$ where $[A, B, C] = [A, [B, C]]$ and $[A, B] = AB - BA$ is the commutator of A and B

— * —

```
--S 60 of 63
comm2(A, B) == A * B - B * A
--R
--R                                                              Type: Void
--E 60


--S 61 of 63
comm3(A, B, C) == comm2(A, comm2(B, C))
--R
--R                                                              Type: Void
--E 61


--S 62 of 63
comm2(A, B)
--R
--R
--R   A is declared as being in SquareMatrix(2,Integer) but has not been
--R      given a value.
--E 62


--S 63 of 63
comm3(A, B, C) + comm3(B, C, A) + comm3(C, A, B)
--R
--R
```

```
--R   A is declared as being in SquareMatrix(2,Integer) but has not been
--R      given a value.
--E 63

)spool


)lisp (bye)
```

—————-

References

[1] nothing